

The discontinuity points set of separately continuous functions on the products of compacts

Mykhaylyuk V.V.

*Chernivtsi National University, Department of Mathematical Analysis,
Kotsjubyns'koho 2, Chernivtsi 58012, Ukraine
vmykhaylyuk@ukr.net*

Abstract

It is solved a problem of construction of separately continuous functions on the product of compacts with a given discontinuity points set. We obtaine the following results.

1. For arbitrary Čech complete spaces X, Y and a separable compact perfect projectively nowhere dense zero set $E \subseteq X \times Y$ there exists a separately continuous function $f : X \times Y \rightarrow \mathbb{R}$ the discontinuity points set of which equals to E .
2. For arbitrary Čech complete spaces X, Y and nowhere dense zero sets $A \subseteq X$ and $B \subseteq Y$ there exists a separately continuous function $f : X \times Y \rightarrow \mathbb{R}$ such that the projections of the discontinuity points set of f coincide with A and B respectively.

An example of Eberlein compacts X, Y and nowhere dense zero sets $A \subseteq X$ and $B \subseteq Y$ such that the discontinuity points set of every separately continuous function $f : X \times Y \rightarrow \mathbb{R}$ does not coincide with $A \times B$, and CH -example of separable Valdivia compacts X, Y and separable nowhere dense zero sets $A \subseteq X$ and $B \subseteq Y$ such that the discontinuity points set of every separately continuous function $f : X \times Y \rightarrow \mathbb{R}$ does not coincide with $A \times B$ are constructed.

Key words: Separately continuous functions, compact space, Eberlein compact, Valdivia compact

AMS Subject Classification: Primary 54C08, 54C30, 54D30

1 Introduction

It follows from Namioka's theorem [1] that for arbitrary compact spaces X, Y and a separately continuous function $f : X \times Y \rightarrow \mathbb{R}$ the set $D(f)$ of discontinuity points of f is a projectively meagre set, that is $D(f) \subseteq A \times B$

where $A \subseteq X$ and $B \subseteq Y$ are meagre sets. In this connection, a problem on a characterization of discontinuity points sets of separately continuous functions on the product of two compact spaces was formulated in [2]. In other words, it is required to establish for what projectively meagre F_σ -set E in the product $X \times Y$ of compact spaces X and Y there exists a separately continuous function $f : X \times Y \rightarrow \mathbb{R}$ with $D(f) = E$? This leads to a solving of the inverse problem of separately continuous mappings theory of the construction of separately continuous function with a given discontinuity points set.

The inverse problem on $[0, 1]^2$ and on the products of metrizable spaces was studied in papers of many mathematicians (W. Young and G. Young, R. Kershner, R. Feiock, Z. Grande, J. Breckenridge and T. Nishiura). The most general result in this direction was obtained in [3]. It gives a characterization of discontinuity points set for separately continuous functions of several variables on the product of spaces each of which is the topological product of separable metrizable factors. This result for function of two compact variables was proved in [4, Theorem 4] and it can be formulated in the following way.

Theorem 1.1 *Let $(X_s : s \in S)$, $(Y_t : t \in T)$ be arbitrary families of metrizable compacts, $X = \prod_{s \in S} X_s$ and $Y = \prod_{t \in T} Y_t$. Then for any set $E \subseteq X \times Y$ the following conditions are equivalent:*

- (i) *there exists a separately continuous function $f : X \times Y \rightarrow \mathbb{R}$ with $D(f) = E$;*
- (ii) *there exists a sequence $(E_n)_{n=1}^\infty$ of projectively nowhere dense zero sets $E_n \subseteq X \times Y$ such that $E = \bigcup_{n=1}^\infty E_n$.*

Recall that a set A in a topological space X is called a *zero set* if there exists a continuous function $f : X \rightarrow [0, 1]$ such that $A = f^{(-1)}(0)$, and a *co-zero set* if $A = X \setminus B$ for some zero set $B \subseteq X$. A set E in the product $X \times Y$ of topological space X and Y is called a *projectively nowhere dense set* if E is contained in the product $A \times B$ of nowhere dense sets $A \subseteq X$ and $B \subseteq Y$.

In other hand, the problem of construction a separately continuous function with a given oscillation was solved in [5]. It follows from [5] that for arbitrary separable projectively meagre F_σ -set E in the product $X \times Y$ of Eberlein compacts X and Y there exists a separately continuous function $f : X \times Y \rightarrow \mathbb{R}$ with $D(f) = E$. Besides, examples of nonseparable closed sets E_1 and E_2 in the products of two Eberlein compacts such that E_1 is the discontinuity points set of some separately continuous function and E_2 is not the discontinuity points set for every separately continuous function on the product of the corresponding spaces.

Note that the following Price-Simon type property of Eberlein compacts (see [6, p.170]) plays an important role in the proof of the results of [5]. For every

Eberlein compact X and $x_0 \in X$ there exists a sequence of nonempty open sets, which converges to x_0 (a sequence $(A_n)_{n=1}^\infty$ of sets $A_n \subseteq Y$ converges to $y_0 \in Y$ in a topological space Y , that is $A_n \rightarrow x_0$, if for every neighbourhood U of y_0 in Y there exists a number $n_0 \in \mathbb{N}$ such that $A_n \subseteq U$ for all $n \geq n_0$).

The problem of construction of separately continuous function on the product of two compact spaces with a given one-point discontinuity points set was solved in [7] using a dependence of functions upon some quantity of coordinates technique. It was obtained in [7] that for nonisolated points x_0 and y_0 in compact spaces X and Y respectively there exists a separately continuous function $f : X \times Y \rightarrow \mathbb{R}$ with $D(f) = \{(x_0, y_0)\}$ if and only if there exist sequences $(U_n)_{n=1}^\infty$ and $(V_n)_{n=1}^\infty$ of nonempty co-zero sets $U_n \subseteq X$ and $V_n \subseteq Y$ which converges to x_0 and y_0 respectively, besides, $x_0 \notin U_n$ and $y_0 \notin V_n$ for every $n \in \mathbb{N}$.

Note that a solving of the inverse problem for a F_σ -set $E = \bigcup_{n=1}^\infty E_n$ is reduces to the construction a separately continuous function f with $D(f) = E_n$ where E_n is a closed set. Therefore the following questions arise naturally in a connection with results mentioned above.

Question 1.2 *Let E be a projectively nowhere dense zero set in the product $X \times Y$ of compacts X and Y . Does there exist a separately continuous function $f : X \times Y \rightarrow \mathbb{R}$ with $D(f)=E$?*

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Besides, theorems on characterizations of discontinuity points sets of separately continuous functions, which were obtained, have been formulated in projections properties terms. Therefore it is important to study a weak inverse problem of construction of a separately continuous function with given projections. It is connected with special inverse problems of construction of separately continuous function with a given discontinuity points set E of special type ($E = A \times B$, $E = \{x_0\} \times \{y_0\}$, etc.), which have been studied in [8, 9]. In particular, the special inverse problem was solved in [8] in the following cases: for a set $A \times \{y_0\}$ where A is any nowhere dense zero set in a topological space X and y_0 is any nonisolated point with a countable base of neighbourhoods in a completely regular space Y ; and for a set $A \times B$ where A and B are nowhere dense zero sets in a topological space X and a locally connected space Y respectively. Thus the following question arises naturally.

Question 1.5 *Let A, B be nowhere dense zero sets in compacts X and Y respectively. Does there exist a separately continuous function $f : X \times Y \rightarrow \mathbb{R}$ such that the projections on X and Y of discontinuity points set of f coincide with A and B respectively?*

In this paper we give positive answers to Question 1.2 if E is a separable perfect set, and to Question 1.5. Further we construct an example which gives a negative answer to Question 1.3 (thus to Question 1.2), and an CH -example which gives a negative answer to Question 1.4.

2 The inverse problem on the product of compacts

Recall some definitions and introduce some notations.

A set A in a topological space X is called a \overline{G}_δ -set if there exists a sequence $(G_n)_{n=1}^\infty$ of open in X sets G_n such that $A = \bigcap_{n=1}^\infty G_n$ and $\overline{G}_{n+1} \subseteq G_n$ for every $n \in \mathbb{N}$ where \overline{B} means the closure of a set B in the corresponding space.

A set A in a topological space X is called a *perfect set* if A is a perfect space in the topology induced by X , that is every closed in A set is a G_δ -set in A .

A function $f : X \rightarrow \mathbb{R}$ defined on a topological space X is called a *lower semi-continuous function at an $x_0 \in X$* if for every $\varepsilon > 0$ there exists a neighbourhood U of x_0 in X such that $f(x) > f(x_0) - \varepsilon$ for any $x \in U$, and a *lower semi-continuous function* if f is lower semi-continuous at any point $x \in X$.

Let X, Y be arbitrary sets. The mappings $pr_X : X \times Y \rightarrow X$ and $pr_Y : X \times Y \rightarrow Y$ are defined as follows: $pr_X(x, y) = x$ and $pr_Y(x, y) = y$ for every $x \in X$ and $y \in Y$. Besides, let $f : X \times Y \rightarrow \mathbb{R}$ be a function. For every $x_0 \in X$ and $y_0 \in Y$ the functions $f^{x_0} : Y \rightarrow \mathbb{R}$ and $f_{y_0} : X \rightarrow \mathbb{R}$ are defined as follows: $f^{x_0}(y) = f(x_0, y)$ and $f_{y_0}(x) = f(x, y_0)$ for any $x \in X$ and $y \in Y$.

Let X be a topological space, $A \subseteq X$ and $f : X \rightarrow \mathbb{R}$. The restriction of f to A we denote by $f|_A$. The real $\omega_f(A) = \sup_{x', x'' \in A} |f(x') - f(x'')|$ is called *the oscillation of f on A* . If $x_0 \in X$ and \mathcal{U} is a system of all neighborhoods of x_0 in X then the real $\omega_f(x_0) = \inf_{U \in \mathcal{U}} \omega_f(U)$ is called *the oscillation of f at x_0* .

For a function $f : X \rightarrow \mathbb{R}$ defined on a set X the set $\text{supp } f = \{x \in X : f(x) \neq 0\}$ is called *a support of f* .

A completely regular space X is called a *Čech complete space* if for every

compactification cX of X the set X is a G_δ -set in cX (see [10, p.297]).

Let X be a topological space. We say that a point $x_0 \in X$ has a weak Price-Simon property in X if there exists a sequence $(U_n)_{n=1}^\infty$ of nonempty open in X sets U_n such that $U_n \rightarrow x_0$, and X has a weak Price-Simon property if every point $x \in X$ has the weak Price-Simon property in X .

The following result take an important place in a solving of the inverse problem and the method of its proof is similar to the method which was used in [11] for the product of separable metrizable spaces.

Theorem 2.1 *Let X, Y be completely regular spaces, $A \subseteq X$ and $B \subseteq Y$ be nowhere dense sets, $E \subseteq A \times B$ be a \overline{G}_δ -set in $Z = X \times Y$ and $P = \{p_n : n \in \mathbb{N}\} \subseteq E$ be a dense in E set such that p_n has the Price-Simon property in Z for every $n \in \mathbb{N}$. Then there exists a lower semi-continuous separately continuous function $f : X \times Y \rightarrow Z$ such that $D(f) = E$.*

Proof. Let $(G_n)_{n=1}^\infty$ be a sequence of open in Z sets such that $E = \bigcap_{n=1}^\infty G_n$ and $\overline{G}_{n+1} \subseteq G_n$ for every $n \in \mathbb{N}$. Since A and B are nowhere dense and each point p_n has the weak Price-Simon property in Z , for every $n \in \mathbb{N}$ there exist sequences $(U_{nk})_{k=1}^\infty$ and $(V_{nk})_{k=1}^\infty$ of nonempty open in X and Y sets U_{nk} and V_{nk} respectively such that $W_{nk} = U_{nk} \times V_{nk} \xrightarrow{k \rightarrow \infty} p_n$, $U_{nk} \cap A = V_{nk} \cap B = \emptyset$ and $W_{nk} \subseteq G_k$ for every $k \in \mathbb{N}$. For every $n, k \in \mathbb{N}$ pick a point $z_{nk} \in W_{nk}$ and a continuous function $f_{nk} : Z \rightarrow [0, 1]$ such that $f_{nk}(z_{nk}) = 1$ and $f_{nk}(z) = 0$ for any $z \in Z \setminus W_{nk}$. Show that the function $f : X \times Y \rightarrow [0, +\infty)$, $f(x, y) = \sum_{n=1}^\infty \sum_{k=n}^\infty f_{nk}(x, y)$, has the desired properties.

For every $n \in \mathbb{N}$ put $W_n = Z \setminus \overline{G}_n$. Since $f_{ik}|_{W_n} = 0$ for all $i \in \mathbb{N}$ and $k \geq n$, $f|_{W_n} = \sum_{i=1}^\infty \sum_{k=i}^\infty f_{ik}|_{W_n} = \sum_{i=1}^n \sum_{k=i}^n f_{ik}|_{W_n}$. Therefore f is continuous at every point of set $\bigcup_{n=1}^\infty W_n = Z \setminus E$.

Besides, since $W_{nk} \cap ((A \times Y) \cup (X \times B)) = \emptyset$ for any $n, k \in \mathbb{N}$, $f^a = f_b = 0$ for any $a \in A$ and $b \in B$. Therefore, in particular, f is a lower semi-continuous separately continuous function.

It remains to show that $E \subseteq D(f)$. Since $f(p_n) = 0$ for each $n \in \mathbb{N}$, $f(z_{nk}) \geq f_{nk}(z_{nk}) = 1$ for each $k \geq n$ and $z_{nk} \xrightarrow{k \rightarrow \infty} p_n$, $p_n \in D(f)$, besides, $\omega_f(p_n) \geq 1$. Using the closeness of $F = \{z \in Z : \omega_f(z) \geq 1\}$ in Z and $F \subseteq D(f)$ we obtain $E = \overline{P} \subseteq F \subseteq D(f)$. \diamond

For a set which is the union of a sequence of zero sets we obtain the following solution to the inverse problem.

Theorem 2.2 *Let X, Y be completely regular spaces, $(E_n)_{n=1}^\infty$ be a sequence of separable projectively nowhere dense \overline{G}_δ -sets E_n in $X \times Y$ and $E = \bigcup_{n=1}^\infty E_n$, besides, every point of E has the weak Price-Simon property in $X \times Y$. Then there exists a separately continuous function $f : X \times Y \rightarrow \mathbb{R}$ such that $D(f) = E$.*

Proof. By Theorem 2.1 for every $n \in \mathbb{N}$ there exists a lower semi-continuous separately continuous function $g_n : X \times Y \rightarrow \mathbb{R}$ such that $D(g_n) = E_n$. Fix any strictly increasing homeomorphism $\varphi : \mathbb{R} \rightarrow (-1, 1)$. Clearly that the functions $f_n : X \times Y \rightarrow (-1, 1)$, $f_n(x, y) = \varphi(g_n(x, y))$, are lower semi-continuous separately continuous and $D(f_n) = E_n$ for every $n \in \mathbb{N}$. By [12, Corollary 2.2.2] for a separately continuous function $f : X \times Y \rightarrow \mathbb{R}$, $f(x, y) = \sum_{n=1}^\infty \frac{1}{2^n} f_n(x, y)$, we have $D(f) = \bigcup_{n=1}^\infty D(f_n) = \bigcup_{n=1}^\infty E_n = E$. \diamond

The following result gives a positive answer to Question 1.2 under some additional conditions on E .

Theorem 2.3 *Let X, Y be Čech complete spaces, $(E_n)_{n=1}^\infty$ be a sequence of separable compact perfect projectively nowhere dense G_δ -sets E_n in $X \times Y$ and $E = \bigcup_{n=1}^\infty E_n$. Then there exists a separately continuous function $f : X \times Y \rightarrow \mathbb{R}$ such that $D(f) = E$.*

Proof. Let \tilde{X}, \tilde{Y} be the Stone-Čech compactifications of X and Y respectively. Since X and Y are Čech complete spaces, X and Y are G_δ -sets in \tilde{X} and \tilde{Y} respectively. Therefore all the sets E_n are G_δ -sets in $\tilde{X} \times \tilde{Y}$. Every one-point subset of E_n is a G_δ -set in the perfect compact E_n . Thus every one-point subset of E is a G_δ -set in the compact $\tilde{X} \times \tilde{Y}$. Hence every point $p \in E$ has a countable base of neighbourhoods in $\tilde{X} \times \tilde{Y}$. Then by Theorem 2.2, there exists a separately continuous function $\tilde{f} : \tilde{X} \times \tilde{Y} \rightarrow \mathbb{R}$ such that $D(\tilde{f}) = E$.

Put $f = \tilde{f}|_{X \times Y}$. Clearly that f is a separately continuous function and $D(f) \subseteq E$. It remains to show that $E \subseteq D(f)$.

Pick a point $p = (x_0, y_0) \in E$, neighbourhoods U and V of x_0 and y_0 in X and Y respectively. Since X and Y are dense in \tilde{X} and \tilde{Y} respectively, $\tilde{U} = \overline{U}$ and $\tilde{V} = \overline{V}$ are neighbourhoods of x_0 and y_0 in \tilde{X} and \tilde{Y} respectively. Using the separate continuity of \tilde{f} we obtain that $\omega_{\tilde{f}}(\tilde{U} \times \tilde{V}) = \omega_{\tilde{f}}(U \times V) = \omega_f(U \times V)$. Therefore $\omega_f(p) = \omega_{\tilde{f}}(p) > 0$ and $p \in D(f)$. \diamond

Note that the Čech completeness of X and Y in Theorem 2.3 cannot be weakened to the complete regularity. Indeed, it was shown in [9, Theorem 1] that an analog of this theorem for completely regular spaces X, Y and one-point set E does not depend of the *ZFC*-axioms.

The method which we use to solving the weak inverse problem is similar to the method from [8]. The following proposition gives a possibility to remove the connection type conditions.

Proposition 2.4 *Let X be a compact space, $A \subseteq X$ be a zero set in X which is not open in X . Then there exists a separately continuous function $f : X \rightarrow [0, 1]$ such that $A = f^{-1}(0)$ and for every open in X set $G \supseteq A$ there exists $n_0 \in \mathbb{N}$ such that $\{\frac{1}{2^n} : n \geq n_0\} \subseteq f(G)$.*

Proof. Let $g : X \rightarrow [0, 1]$ be a continuous function such that $A = g^{-1}(0)$. Since A is not open in X , $g^{-1}([0, \varepsilon)) \setminus A \neq \emptyset$ for every $\varepsilon > 0$. Therefore there exists a sequence $(x_n)_{n=1}^\infty$ of points $x_n \in X$ such that $g(x_{n+1}) < g(x_n) < 1$ for every $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} g(x_n) = 0$. Pick any strictly increasing continuous function $\varphi : [0, 1] \rightarrow [0, 1]$ such that $\varphi(g(x_n)) = \frac{1}{2^n}$. Put $f(x) = \varphi(g(x))$ for every $x \in X$. Clearly that $f : X \rightarrow [0, 1]$ is a continuous function and $A = f^{-1}(0)$. For every $n \in \mathbb{N}$ put $G_n = f^{-1}((\frac{1}{2^n}, 1])$. Let G be an arbitrary open in X set with $A \subseteq G$. Choosing a finite subcover from the open cover $\{G\} \cup \{G_n : n \in \mathbb{N}\}$ of compact space X we obtain an $n_0 \in \mathbb{N}$ such that $G \cup G_{n_0} = X$. Since $f(x_n) = \frac{1}{2^n} \leq \frac{1}{2^{n_0}}$ for every $n \geq n_0$, $\{\frac{1}{2^n} : n \geq n_0\} = \{f(x_n) : n \geq n_0\} \subseteq G$. \diamond

The following theorem gives a positive answer to Question 1.5.

Theorem 2.5 *Let X, Y be Čech complete spaces, $(A_n)_{n=1}^\infty, (B_n)_{n=1}^\infty$ be sequences of nowhere dense compact G_δ -sets A_n and B_n in X and Y respectively, $A = \bigcup_{n=1}^\infty A_n$ and $B = \bigcup_{n=1}^\infty B_n$. Then there exists a separately continuous function $f : X \times Y \rightarrow \mathbb{R}$ such that $\text{pr}_X D(f) = A$ and $\text{pr}_Y D(f) = B$.*

Proof. Note that it is sufficiently to prove this theorem for nowhere dense compact G_δ -sets A and B in compacts X and Y respectively and a lower semi-continuous separately continuous function f analogously as in the proof of Theorem 2.3.

Since A and B are zero sets in X and Y respectively, by Proposition 2.4 there exist continuous functions $g : X \rightarrow [0, 1]$ and $h : Y \rightarrow [0, 1]$ such that $A = g^{-1}(0)$, $B = h^{-1}(0)$ and for every open sets $G_1 \supseteq A$ and $G_2 \supseteq B$ in X and Y respectively there exists an $n_0 \in \mathbb{N}$ such that $\{\frac{1}{2^n} : n \geq n_0\} \subseteq g(G_1)$ and $\{\frac{1}{2^n} : n \geq n_0\} \subseteq h(G_2)$.

Consider the function

$$f(x, y) = \begin{cases} \frac{2g(x)h(y)}{g^2(x)+h^2(y)}, & \text{if } (x, y) \notin A \times B; \\ 0, & \text{if } (x, y) \in A \times B. \end{cases}$$

It is easy to see that f is a lower semi-continuous separately continuous func-

tion and $D(f) \subseteq A \times B$.

Suppose that $\text{pr}_X D(f) \neq A$, that is, there exists an $x_0 \in A \setminus \text{pr}_X D(f)$. Since f is continuous at every point of the compact set $\{x_0\} \times B$ and $f(x_0, y) = 0$ for every $y \in B$, there exists a neighbourhood U of x_0 in X and an open in Y set G such that $f(x, y) < \frac{4}{5}$ for any $x \in U$ and $y \in G$. It follows from the choice of h that there exists an $n_0 \in \mathbb{N}$ such that $\{\frac{1}{2^n} : n \geq n_0\} \subseteq h(G)$. Since $A = g^{-1}(0)$ is nowhere dense, $(g^{-1}([0, \frac{1}{2^{n_0}})) \cap U) \setminus A \neq \emptyset$, that is there exists an $x_1 \in U$ such that $g(x_1) \in (0, \frac{1}{2^{n_0}})$. Choose an $n \geq n_0$ and a $y_1 \in G$ such that $g(x_1) \in [\frac{1}{2^{n+1}}, \frac{1}{2^n})$ and $h(y_1) = \frac{1}{2^n}$. Then

$$f(x_1, y_1) = \frac{2g(x_1)h(y_1)}{g^2(x_1) + h^2(y_1)} \geq \frac{2\frac{1}{2^{n+1}}\frac{1}{2^n}}{\frac{1}{4^{n+1}} + \frac{1}{4^n}} = \frac{4}{5},$$

but it contradicts the choice of U and G .

The equality $\text{pr}_Y D(f) = B$ can be obtained analogously. \diamond

The same reasoning as after the proof of Theorem 2.3 shows that the Čech completeness of X and Y in Theorem 2.5 cannot be weakened to the complete regularity.

3 Separately continuous functions on the product of Eberlein compacts

In this section we construct an example which gives a negative answer to Question 1.3.

Recall that a compact space X which is homeomorphic to some weakly compact subset of a Banach space is called *an Eberlein compact*. The Amir-Lindenstrauss theorem [13] states that a compact X is an Eberlein compact if and only if it is homeomorphic to some compact subset of space $c_0(T)$ ($c_0(T)$ is the space of all functions $x : T \rightarrow \mathbb{R}$ such that for every $\varepsilon > 0$ the set $\{t \in T : |x(t)| \geq \varepsilon\}$ is finite with the topology of pointwise convergence on T).

An idea of the corresponding space construction is closely related to the following simple fact.

Proposition 3.1 *Let $f : [0, 1]^2 \rightarrow \mathbb{R}$ be a separately continuous function. Then there exist strictly decreasing sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ of reals $a_n, b_n \in (0, 1]$ such that $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = 0$ and $|f(a_n, b_m) - f(0, 0)| < \frac{1}{\min\{n, m\}}$ for every $n, m \in \mathbb{N}$.*

Proof. Since f_0 is continuous at 0, there exists an $a_1 \in (0, 1)$ such that

$$|f(a_1, 0) - f(0, 0)| < \frac{1}{2}.$$

Using the continuity of f^0 and f^{a_1} at 0 we choose $b_1 \in (0, 1)$ such that

$$|f(0, b_1) - f(0, 0)| < \frac{1}{2} \quad \text{and} \quad |f(a_1, y) - f(a_1, 0)| < \frac{1}{2}$$

for every $y \in [0, b_1]$. Further, using the continuity of f_0 and f_{b_1} at 0 choose an $a_2 \in (0, \min\{\frac{1}{2}, a_1\})$ so that

$$|f(a_2, 0) - f(0, 0)| < \frac{1}{4} \quad \text{and} \quad |f(x, b_1) - f(0, b_1)| < \frac{1}{2}$$

for every $x \in [0, a_2]$. Since f^0 and f^{a_2} are continuous at 0, there exists $b_2 \in (0, \min\{\frac{1}{2}, b_1\})$ such that

$$|f(0, b_2) - f(0, 0)| < \frac{1}{4} \quad \text{and} \quad |f(a_2, y) - f(a_2, 0)| < \frac{1}{4}$$

for every $y \in [0, b_2]$.

Continuing this procedure to infinity we obtain strictly decreasing sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ of reals $a_n, b_n \in (0, 1]$ such that $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = 0$ and

$$|f(a_n, 0) - f(0, 0)| < \frac{1}{2n}, \quad |f(0, b_n) - f(0, 0)| < \frac{1}{2n},$$

$$|f(a_n, y) - f(a_n, 0)| < \frac{1}{2n} \quad \text{and} \quad |f(x, b_n) - f(0, b_n)| < \frac{1}{2n}$$

for every $y \in [0, b_n]$ and $x \in [0, a_{n+1}]$. Then for $m \geq n$ we have

$$\begin{aligned} |f(a_n, b_m) - f(0, 0)| &\leq |f(a_n, b_m) - f(a_n, 0)| + |f(a_n, 0) - f(0, 0)| < \\ &< \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}. \end{aligned}$$

And for $n > m$ we have

$$\begin{aligned} |f(a_n, b_m) - f(0, 0)| &\leq |f(a_n, b_m) - f(0, b_m)| + |f(0, b_m) - f(0, 0)| < \\ &< \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m}. \end{aligned}$$

◇

For the topological product $X = \prod_{s \in S} X_s$ of a family $(X_s : s \in S)$ of topological spaces X_s and a nonempty basic open set $U = \prod_{s \in S} U_s$ put $R(U) = \{s \in S : U_s \neq X_s\}$. Let, besides, Y be a subspace of X . An nonempty open in Y set

V is called a *basic open set* if there exists a basic open set $U = \varphi(V)$ in X such that $V = U \cap Y$. For any nonempty basic open set V in Y we put $R(V) = R(\varphi(V))$.

The following theorem is a main result of this section.

Theorem 3.2 *There exist Eberlein compacts X and Y and nowhere dense zero sets A and B in X and Y respectively such that $D(f) \neq A \times B$ for every separately continuous function $f : X \times Y \rightarrow \mathbb{R}$.*

Proof. Denote the set of all strictly decreasing sequences $s = (\alpha_n)_{n=1}^\infty$ of reals $\alpha_n \in (0, 1]$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ by S_0 and $S = \{0\} \cup S_0$. For every $s = (\alpha_n) \in S_0$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the function $x(s, n) \in [0, 1]^S$ is defined as follows: if $n \in \mathbb{N}$, then

$$x(s, n)(t) = \begin{cases} 1, & \text{if } t = s, \\ 0, & \text{if } t \in S_0 \setminus \{s\}, \\ \alpha_n, & \text{if } t = 0, \end{cases}$$

and

$$x(s, 0)(t) = \begin{cases} 1, & \text{if } t = s, \\ 0, & \text{if } t \in S \setminus \{s\}. \end{cases}$$

Put $X_0 = [0, 1] \times \{0\}^{S_0}$, $X_s = \{x(s, n) : n \in \mathbb{N}_0\}$ for every $s \in S_0$ and $X = \bigcup_{s \in S} X_s$. Show that X is a closed subspace of $Z = [0, 1] \times \{0, 1\}^{S_0}$.

Since for every $x \in X$ the set $\{s \in S_0 : x(s) = 1\}$ has at most one element, all functions $z \in \overline{X}$ have the same properties. Therefore it is sufficient to prove that for every $z \in Z \setminus X$ with $|\{s \in S_0 : z(s) = 1\}| \leq 1$ there exists an open neighbourhood U of z in Z such that $U \cap X = \emptyset$.

Pick $z_0 \in Z \setminus X$ such that $|\{s \in S_0 : z(s) = 1\}| \leq 1$. Note that $|\{s \in S_0 : z(s) = 1\}| = 1$. Indeed, if $z_0(s) = 0$ for every $s \in S_0$ then $z_0 \in X_0$ which contradicts the choice of z_0 . Pick $s = (\alpha_n) \in S_0$ such that $z_0(s) = 1$. Since $z_0 \neq x(s, 0)$, $z_0(0) > 0$. Note $z_0 \neq x(s, n)$ for every $n \in \mathbb{N}$, therefore $z_0(0) \neq \alpha_n$ for every $n \in \mathbb{N}$. Choose an open neighbourhood I of $z_0(0)$ in $[0, 1]$ so that $0 \notin I$ and $\alpha_n \notin I$ for every $n \in \mathbb{N}$. For the open neighbourhood $U = \{z \in Z : z(0) \in I, z(s) = 1\}$ of z_0 in Z we have $U \cap X = \emptyset$.

Thus X is a compact. Since the supports of all functions $x \in X$ are finite, X is an Eberlein compact by [13].

Put $A = \{x \in X : x(0) = 0\}$. Clearly that A is a zero set in X . Since

$X_s = \overline{X_s \setminus A}$ for every $s \in S$, $X \setminus A$ is a dense in X set. Therefore A is a nowhere dense in X set.

Denote $Y = X$, $B = A$ and suppose that there exists a separately continuous function $f : X \times Y \rightarrow \mathbb{R}$ such that $D(f) = A \times B$. Note that the function $\varphi : X_0 \rightarrow [0, 1]$, $\varphi(x) = x(0)$, is a homeomorphism, therefore the function $g : [0, 1]^2 \rightarrow \mathbb{R}$, $g(u, v) = f(\varphi^{-1}(u), \varphi^{-1}(v))$, is separately continuous. By Proposition 3.1 there exist strictly decreasing sequences $(u_n)_{n=1}^\infty$ and $(v_n)_{n=1}^\infty$ of reals $u_n, v_n \in (0, 1]$ such that $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} v_n = 0$ and $|g(u_n, v_m) - g(0, 0)| < \frac{1}{\min\{n, m\}}$ for every $n, m \in \mathbb{N}$.

For every $n \in \mathbb{N}$ put $x_n = \varphi^{-1}(u_n)$ and $y_n = \varphi^{-1}(v_n)$. Since for every $n, m \in \mathbb{N}$ f is a jointly continuous function at (x_n, y_m) , there exist basic open neighbourhoods U_{nm} and V_{nm} of x_n and y_m in X and Y respectively such that $|f(x, y) - f(x_n, y_m)| < \frac{1}{\min\{n, m\}}$ for every $x \in U_{nm}$ and $y \in V_{nm}$.

Consider an at most countable set $T = \bigcup_{n, m=1}^\infty (R(U_{nm}) \cup R(V_{nm}))$. Since the set of all subsequences of some sequence has the cardinality 2^{\aleph_0} , there exists an increasing sequence $(n_k)_{k=1}^\infty$ of $n_k \in \mathbb{N}$ such that $s = (\alpha_k)_{k=1}^\infty$, $t = (\beta_k)_{k=1}^\infty \notin T$, where $\alpha_k = u_{n_k}$ and $\beta_k = v_{n_k}$ for every $k \in \mathbb{N}$. Note that $x_0 = x(s, 0) \in A$ and $y_0 = x(t, 0) \in B$. Show that f is continuous at (x_0, y_0) , which is impossible.

Fix $\varepsilon > 0$ and choose a number k_0 such that $\frac{1}{n_{k_0}} < \frac{\varepsilon}{2}$. Note that the sets $U = \{x \in X : x(s) = 1, x(0) \leq \alpha_{k_0}\} = \{x(s, k) : k = 0 \text{ or } k \geq k_0\}$ and $V = \{y \in Y : y(t) = 1, y(0) \leq \beta_{k_0}\} = \{x(t, k) : k = 0 \text{ or } k \geq k_0\}$ are neighbourhoods of x_0 and y_0 in X and Y respectively. Pick $i, j \geq k_0$. It follows from $s \notin R(U_{n_i n_j})$, $t \notin R(V_{n_i n_j})$, $x_{n_i}(0) = \alpha_i$ and $y_{n_j}(0) = \beta_j$, that $x(s, i) \in U_{n_i n_j}$ and $x(t, j) \in V_{n_i n_j}$. Therefore $|f(x(s, i), x(t, j)) - f(x_{n_i}, y_{n_j})| < \frac{1}{\min\{n_i, n_j\}}$. It follows from $f(x_{n_i}, y_{n_j}) = g(u_{n_i}, v_{n_j})$ and the choosing of sequences $(u_n)_{n=1}^\infty$ and $(v_n)_{n=1}^\infty$ that

$$|f(x(s, i), x(t, j)) - g(0, 0)| < \frac{2}{\min\{n_i, n_j\}}.$$

Since f is a separately continuous function, $x_0 = \lim_{n \rightarrow \infty} x(s, n)$ and $y_0 = \lim_{n \rightarrow \infty} x(t, n)$, $|f(x(s, i), y_0) - g(0, 0)| \leq \frac{2}{n_i}$, $|f(x_0, x(t, j)) - g(0, 0)| \leq \frac{2}{n_j}$ and $f(x_0, y_0) = g(0, 0)$. Thus $|f(x, y) - f(x_0, y_0)| \leq \frac{2}{n_{k_0}} < \varepsilon$ for every $(x, y) \in U \times V$.

◇

4 Separately continuous functions on the products of separable Valdivia compacts

Recall that a compact space X is called a *Corson compact* if it is homeomorphic to a compact $Z \subseteq \mathbb{R}^T$ such that $|\text{supp } z| \leq \aleph_0$ for every $z \in Z$, and a *Valdivia compact* if it is homeomorphic to a compact $Z \subseteq \mathbb{R}^T$ such that the set $\{z \in Z : |\text{supp } z| \leq \aleph_0\}$ is dense in Z . Clearly that any Corson compact is a Valdivia compact. Besides, it follows from [13] that any Eberlein compact is a Corson compact.

Since every separable subset of a Corson compact is metrizable, it follows from Theorem 2.3 that Question 1.4 has a positive answer for Corson compacts. Therefore it is naturally to establish whether is it true for Valdivia compacts.

In this section we show that in CH -assumption Question 1.4 has a negative answer even for separable Valdivia compacts.

The following notation is an important tool for the construction of the corresponding example.

Let $X \subseteq [0, 1]^S$, $Y \subseteq [0, 1]^T$ be arbitrary spaces, $s_0 \in S$, $t_0 \in T$ and $f : X \times Y \rightarrow \mathbb{R}$ be a function. We say that sequences $(u_n)_{n=1}^\infty$ and $(v_n)_{n=1}^\infty$ of reals $u_n, v_n \in (0, 1]$ *nullify f in the coordinates s_0 and t_0* if the following conditions hold:

- (1_n) $|f(x, y_1) - f(x, y_2)| < \frac{1}{n}$
for every $n \in \mathbb{N}$, $x \in X$ with $x(s_0) = u_n$, $m \geq n$, $y_1, y_2 \in Y$ with $y_1(t) = y_2(t)$ for $t \in T \setminus \{t_0\}$, $y_1(t_0) = v_m$ and $y_2(t_0) = 0$;
- (2_m) $|f(x_1, y) - f(x_2, y)| < \frac{1}{m}$
for every $m \in \mathbb{N}$, $y \in Y$ with $y(t_0) = v_m$, $n > m$, $x_1, x_2 \in X$ with $x_1(s) = x_2(s)$ for $s \in S \setminus \{s_0\}$, $x_1(s_0) = u_n$ and $x_2(s_0) = 0$.

Proposition 4.1 *Let $X \subseteq [0, 1]^S$, $Y \subseteq [0, 1]^T$ be compacts, $s_0 \in S$, $t_0 \in T$, $A = \{x \in X : x(s_0) = 0\}$, $B = \{y \in Y : y(t_0) = 0\}$, $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ be strictly decreasing sequences of reals $a_n, b_n \in (0, 1]$ with $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = 0$ and $f : X \times Y \rightarrow \mathbb{R}$ be a function with $D(f) \subseteq A \times B$. Then there exist subsequences $(u_n)_{n=1}^\infty$ and $(v_n)_{n=1}^\infty$ of sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ respectively which nullify f in the coordinates s_0 and t_0 .*

Proof. For every $n \in \mathbb{N}$ put $A_n = \{x \in X : x(s_0) = a_n\}$ and $B_n = \{y \in Y : y(t_0) = b_n\}$. Since f is jointly continuous at any point of compacts $A_n \times Y$, for every $n \in \mathbb{N}$ there exists $\varepsilon_n > 0$ such that

$$|f(x, y_1) - f(x, y_2)| < \frac{1}{n}$$

for any $x \in A_n$, $y_1, y_2 \in Y$ with $y_1(t) = y_2(t)$ for $t \in T \setminus \{t_0\}$ and $|y_1(t_0) - y_2(t_0)| < \varepsilon_n$.

Analogously, for every $m \in \mathbb{N}$ the joint continuity of f on the compact $X \times B_m$ implies the existence of a $\delta_m > 0$ such that

$$|f(x_1, y) - f(x_2, y)| < \frac{1}{m}$$

for any $y \in B_m$, $x_1, x_2 \in X$ with $x_1(s) = x_2(s)$ for $s \in S \setminus \{s_0\}$ and $|x_1(s_0) - x_2(s_0)| < \delta_m$.

Denote $i_1 = 1$ and choose strictly increasing sequences $(i_n)_{n=2}^\infty$ and $(j_n)_{n=1}^\infty$ of numbers $i_n, j_n \in \mathbb{N}$ such that $b_{j_n} < \varepsilon_{i_n}$ and $a_{i_{n+1}} < \delta_{j_n}$ for every $n \in \mathbb{N}$.

It remains to put $u_n = a_{i_n}$ and $v_n = b_{j_n}$ for every $n \in \mathbb{N}$. \diamond

Now describe a method of the construction of Valdivia compacts.

Let \mathcal{A} be a system of sets $A \subseteq [0, 1]$, $S = \{0\} \cup \mathcal{A}$, $X_0 = [0, 1]$, $X_s = \{0, 1\}$ for every $s \in \mathcal{A}$ and $X = \prod_{s \in S} X_s$. For every finite set $T \subseteq \mathcal{A}$ put

$$Z_T = \{x \in X : x(s) = 1 \forall s \in T, x(s) = 0 \forall s \in \mathcal{A} \setminus T, x(0) \in \bigcap_{A \in T} A\},$$

if $T \neq \emptyset$, and

$$Z_\emptyset = \{x \in X : x(s) = 0 \forall s \in \mathcal{A}, x(0) \in \bigcup \mathcal{A}\}.$$

The compact subspace $X_{\mathcal{A}} = \overline{\bigcup \{Z_T : T \subseteq \mathcal{A}, T \text{ is finite}\}}$ of the space X is called a *compact generated by the system \mathcal{A}* . Clearly that $X_{\mathcal{A}}$ is a Valdivia compact.

We use the following properties of compacts generated by systems.

Proposition 4.2 *Let \mathcal{A} be a system of sets $A \subseteq [0, 1)$, $X = X_{\mathcal{A}}$ and $s_0 = A_0 \in \mathcal{A}$. Then for every $x \in X_{\mathcal{A}}$ if $x(s_0) = 1$ then $x(0) \in \overline{A_0}$.*

Proof. It follows from the definition of $X_{\mathcal{A}}$ that for every $x \in \bigcup \{Z_T : T \subseteq \mathcal{A}, T \text{ is finite}\}$ if $x(s_0) = 1$ then $x(0) \in A_0$. It remains to apply the closure operation. \diamond

Proposition 4.3 *Let \mathcal{A} be a system of sets $A \subseteq [0, 1)$ such that the set $A_0 = \bigcup \mathcal{A}$ is at most countable. Then $X_{\mathcal{A}}$ is a separable compact.*

Proof. Let $A_0 = \{a_n : n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$ put $\mathcal{A}_n = \{A \in \mathcal{A} : a_n \in A\}$ and $X_n = \{x \in X_{\mathcal{A}} : x(0) = a_n, x(s) = 0 \forall s \in \mathcal{A} \setminus \mathcal{A}_n\}$. Note that for every

finite set $T \subseteq \mathcal{A}_n$ the function

$$x_T(s) = \begin{cases} a_n, & \text{if } s = 0; \\ 1, & \text{if } s \in T; \\ 0, & \text{if } s \in \mathcal{A} \setminus T, \end{cases}$$

belongs to $X_{\mathcal{A}}$. Therefore $X_n = \{a_n\} \times \prod_{s \in \mathcal{A}_n} \{0, 1\} \times \prod_{s \in \mathcal{A} \setminus \mathcal{A}_n} \{0\}$. Since $|\mathcal{A}_n| \leq 2^{\aleph_0}$, X_n is a separable space by Hewitt-Marczewski-Pondiczery theorem [10, p.133].

Since $Z_T \subseteq \bigcup_{n=1}^{\infty} X_n$ for any finite set $T \subseteq \mathcal{A}$, $X_{\mathcal{A}} = \overline{\bigcup_{n=1}^{\infty} X_n}$. Thus $X_{\mathcal{A}}$ is a separable space. \diamond

Proposition 4.4 *Let \mathcal{A} be a system of sets $A \subseteq [0, 1)$ and $\mathcal{B} \subseteq \mathcal{A}$ such that $\bigcup \mathcal{B} = \bigcup \mathcal{A}$. Then $\varphi(X_{\mathcal{A}}) = X_{\mathcal{B}}$ where $\varphi : X_{\mathcal{A}} \rightarrow \mathbb{R}^{\{0\} \cup \mathcal{B}}$, $\varphi(x) = x|_{\{0\} \cup \mathcal{B}}$.*

Proof. The inclusion $X_{\mathcal{B}} \subseteq \varphi(X_{\mathcal{A}})$ follows immediately from the definition of a compact generated by a system.

Fix a finite subsystem $T \subseteq \mathcal{A}$. If $T \cap \mathcal{B} = \emptyset$, then $\bigcup \mathcal{B} = \bigcup \mathcal{A}$ implies $\varphi(Z_T) \subseteq X_{\mathcal{B}}$. If $T \cap \mathcal{B} \neq \emptyset$, then the inclusion $\varphi(Z_T) \subseteq X_{\mathcal{B}}$ follows from the definition of Z_T .

Since the set $\bigcup \{\varphi(Z_T) : T \subseteq \mathcal{A}, T \text{ is finite}\}$ is dense in $\varphi(X_{\mathcal{A}})$, $X_{\mathcal{B}}$ is dense in $\varphi(X_{\mathcal{A}})$. Thus $\varphi(X_{\mathcal{A}}) \subseteq X_{\mathcal{B}}$. \diamond

We use also the following two facts.

Proposition 4.5 *Let $(A_n)_{n=1}^{\infty}$ be a sequence of infinite sets $A_n \subseteq \mathbb{N}$ such that $\bigcap_{k=1}^n A_k$ is an infinite set for every $n \in \mathbb{N}$. Then there exists an infinite set $B \subseteq \mathbb{N}$ such that $|B \setminus A_n| < \aleph_0$ for every $n \in \mathbb{N}$.*

Proof. It is sufficient to put $n_1 = \min A_1$, $n_k = \min(\bigcap_{i=1}^{k+1} A_i \setminus \{n_1, \dots, n_{k-1}\})$ for every $k \geq 2$ and $B = \{n_k : k \in \mathbb{N}\}$. \diamond

Proposition 4.6 *Let ω be the first ordinal of some infinite cardinality. Then there exists a bijection $\varphi : [1, \omega)^2 \rightarrow [1, \omega)$ such that $\varphi(\xi, \eta) \geq \xi$ for every $\xi, \eta \in [1, \omega)$.*

Proof. Note that $|[1, \omega)^2| = |[1, \omega)|$ by Hessenberg's theorem [14, p. 284], that is, there exists a bijection $\psi : [1, \omega) \rightarrow [1, \omega)^2$. For every $\xi \in [1, \omega)$ denote $(\alpha_{\xi}, \beta_{\xi}) = \psi(\xi)$.

Using the transfinite induction we construct a bijection $\tilde{\varphi} : [1, \omega) \rightarrow [1, \omega)$

such that $\tilde{\varphi}(\xi) \geq \alpha_\xi$ for every $\xi \in [1, \omega)$.

Put $\tilde{\varphi}(1) = \alpha_1$.

Assume that $\tilde{\varphi}(\eta)$ is defined for all $\eta \in [1, \xi)$ where $\xi \in (1, \omega)$. Put $\tilde{\varphi}(\xi) = \min([\alpha_\xi, \omega) \setminus \{\tilde{\varphi}(\eta) : 1 \leq \eta < \xi\})$.

Clearly that $\tilde{\varphi}$ is an injection and $\tilde{\varphi}(\xi) \geq \alpha_\xi$ for every $\xi \in [1, \omega)$. Show that $\tilde{\varphi}$ is a surjection.

Fix a $\xi \in [1, \omega)$. Choose an $\eta \in [1, \omega)$ such that $\psi(\eta) = (\xi, 1)$, that is $a_\eta = \xi$ and $b_\eta = 1$. If $\tilde{\varphi}(\eta) \neq \xi$, then $\xi \in \{\tilde{\varphi}(\zeta) : 1 \leq \zeta \leq \eta\}$.

It remains to put $\varphi = \tilde{\varphi} \circ \psi^{-1}$. ◇

Let Z, S be arbitrary sets, $X \subseteq \mathbb{R}^S$ and $f : X \rightarrow Z$. We say that f *depends upon a countable quantity of coordinates* if there exists an at most countable set $T \subseteq S$ such that $f(x') = f(x'')$ for every $x', x'' \in X$ with $x'|_T = x''|_T$. It is easy to see that for any compact $X \subseteq \mathbb{R}^S$ every continuous function $f : X \rightarrow \mathbb{R}$ depends upon a countable quantity of coordinates. It follows from [7, Theorem 1] that if $X \subseteq \mathbb{R}^S$ and $Y \subseteq \mathbb{R}^T$ are separable compacts, then every separately continuous function $f : X \times Y \rightarrow \mathbb{R}$ depends upon countable quantity of coordinates as a mapping defined on $X \times Y$, that is there exist at most countable sets $S_0 \subseteq S$ and $T_0 \subseteq T$ such that $f(x', y') = f(x'', y'')$ for every $x', x'' \in X$ with $x'|_{S_0} = x''|_{S_0}$ and $y', y'' \in Y$ with $y'|_{T_0} = y''|_{T_0}$.

The following theorem is the main result of this section.

Theorem 4.7 (CH) *There exist separable Valdivia compacts X and Y , nowhere dense separable zero sets E and F in X and Y respectively such that $D(f) \neq E \times F$ for every separately continuous function $f : X \times Y \rightarrow \mathbb{R}$.*

Proof. Put $A_0 = B_0 = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. Using the transfinite induction we construct families $(A_\xi : 1 \leq \xi < \omega_1)$ and $(B_\xi : 1 \leq \xi < \omega_1)$ of sets $A_\xi = \{0\} \cup \{a_n^{(\xi)} : n \in \mathbb{N}\} \subseteq A_0$ and $B_\xi = \{0\} \cup \{b_n^{(\xi)} : n \in \mathbb{N}\} \subseteq B_0$ where $(a_n^{(\xi)})_{n=1}^\infty$ and $(b_n^{(\xi)})_{n=1}^\infty$ are strictly decreasing sequences which satisfy the following conditions:

- (1) $A_\xi \setminus A_\eta$ and $B_\xi \setminus B_\eta$ are finite sets for every $0 \leq \eta < \xi < \omega_1$;
- (2) for every $\xi \in [1, \omega_1)$ and separately continuous function $g : X_{\mathcal{A}_\xi} \times X_{\mathcal{B}_\xi} \rightarrow \mathbb{R}$ with $D(g) \subseteq E_\xi \times F_\xi$, where $\mathcal{A}_\xi = \{A_\zeta : 0 \leq \zeta < \xi\}$, $\mathcal{B}_\xi = \{B_\zeta : 0 \leq \zeta < \xi\}$, $E_\xi = \{x \in X_{\mathcal{A}_\xi} : x(0) = 0\}$ and $F_\xi = \{y \in X_{\mathcal{B}_\xi} : y(0) = 0\}$, there exists an $\eta \in [1, \omega_1)$ such that the sequences $(a_n^{(\eta)})_{n=1}^\infty$ and $(b_n^{(\eta)})_{n=1}^\infty$ nullify g in the coordinates $s_0 = 0$ and $t_0 = 0$.

Using Proposition 4.6 choose a bijection

$$[1, \omega_1) \ni \xi \xrightarrow{\varphi} (\varphi_1(\xi), \varphi_2(\xi)) \in [1, \omega_1)^2$$

so that $\varphi_1(\xi) \leq \xi$ for every $\xi \in [1, \omega_1)$, in particular, $\varphi_1(1) = 1$.

Since $X_{\mathcal{A}_1}$ and $X_{\mathcal{B}_1}$ are separable by Proposition 4.3, every separately continuous function $g : X_{\mathcal{A}_1} \times X_{\mathcal{B}_1} \rightarrow \mathbb{R}$ is determined by its values on some at most countable dense subset of $X_{\mathcal{A}_1} \times X_{\mathcal{B}_1}$. Therefore the system \mathcal{F}_1 of all separately continuous functions $g : X_{\mathcal{A}_1} \times X_{\mathcal{B}_1} \rightarrow \mathbb{R}$ with $D(g) \subseteq E_1 \times F_1$ has the cardinality 2^{\aleph_0} , that is $\mathcal{F}_1 = \{g_{(1, \eta)} : 1 \leq \eta < \omega_1\}$. Using Proposition 4.1 choose subsequences $(a_n^{(1)})_{n=1}^\infty$ and $(b_n^{(1)})_{n=1}^\infty$ of the sequence $(\frac{1}{n})_{n=1}^\infty$ which nullify $g_{\varphi(1)}$ in the coordinates $s_0 = 0$ and $t_0 = 0$.

Assume that the sets A_η and B_η for $1 \leq \eta < \xi < \omega_1$ are constructed such that condition (1) holds and for every $\eta \in [1, \xi)$ the sequences $(a_n^{(\eta)})_{n=1}^\infty$ and $(b_n^{(\eta)})_{n=1}^\infty$ nullify $g_{\varphi(\eta)}$ in the coordinates $s_0 = 0$ and $t_0 = 0$ where $\mathcal{F}_\eta = \{g_{(\eta, \zeta)} : 1 \leq \zeta < \omega_1\}$ is the system of all separately continuous functions $g : X_{\mathcal{A}_\eta} \times X_{\mathcal{B}_\eta} \rightarrow \mathbb{R}$ with $D(g) \subseteq E_\eta \times F_\eta$.

It follows from Proposition 4.3 that $X_{\mathcal{A}_\xi}$ and $X_{\mathcal{B}_\xi}$ are separable. Therefore the system \mathcal{F}_ξ of all separately continuous functions $g : X_{\mathcal{A}_\xi} \times X_{\mathcal{B}_\xi} \rightarrow \mathbb{R}$ with $D(g) \subseteq E_\xi \times F_\xi$ has the cardinality 2^{\aleph_0} , that is $\mathcal{F}_\xi = \{g_{(\xi, \eta)} : 1 \leq \eta < \omega_1\}$.

Besides, since $\varphi_1(\xi) \leq \xi$, we have $g_{\varphi(\xi)} \in \bigcup_{\eta=1}^{\xi} \mathcal{F}_\eta$. Using Proposition 4.5 choose strictly decreasing sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ of reals $a_n, b_n \in A_0$ such that for every $\eta \in [1, \xi)$ the sets $\{a_n : n \in \mathbb{N}\} \setminus A_\eta$ and $\{b_n : n \in \mathbb{N}\} \setminus B_\eta$ are finite. Now using Proposition 4.1 choose subsequences $(a_n^{(\xi)})_{n=1}^\infty$ and $(b_n^{(\xi)})_{n=1}^\infty$ of sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ respectively which nullify $g_{\varphi(\xi)}$ in the coordinates $s_0 = 0$ and $t_0 = 0$.

Clearly that the families $(A_\xi : 1 \leq \xi < \omega_1)$ and $(B_\xi : 1 \leq \xi < \omega_1)$ satisfy (1). Show that the condition (2) holds.

Fix $\xi \in [1, \omega_1)$ and $g \in \mathcal{F}_\xi$. Then there exists a $\zeta \in [1, \omega_1)$ such that $g = g_{(\xi, \zeta)}$. Since φ is a bijection, there exists $\eta \in [1, \omega_1)$ such that $\varphi(\eta) = (\xi, \zeta)$. Then the sequences $(a_n^{(\eta)})_{n=1}^\infty$ and $(b_n^{(\eta)})_{n=1}^\infty$ nullify $g_{(\xi, \zeta)}$ in the coordinates $s_0 = 0$ and $t_0 = 0$.

Put $\mathcal{A} = \{A_\xi : 0 \leq \xi < \omega_1\}$, $\mathcal{B} = \{B_\xi : 0 \leq \xi < \omega_1\}$, $X = X_{\mathcal{A}}$, $Y = Y_{\mathcal{B}}$, $E = \{x \in X : x(0) = 0\}$ and $F = \{y \in Y : y(0) = 0\}$. Note that compacts E and F are homeomorphic to $\{0, 1\}^{\omega_1}$. Therefore E and F are separable. It easy to see that E and F are nowhere dense in X and Y respectively. Besides, by Proposition 4.4 for every $\xi \in [1, \omega_1)$ we have $\pi_1^{(\xi)}(X) = X_{\mathcal{A}_\xi}$ and $\pi_2^{(\xi)}(Y) = X_{\mathcal{B}_\xi}$ where $\pi_1^{(\xi)} : X \rightarrow \mathbb{R}^{\{0\} \cup \mathcal{A}_\xi}$, $\pi_1^{(\xi)}(x) = x|_{\{0\} \cup \mathcal{A}_\xi}$, and $\pi_2^{(\xi)} : Y \rightarrow \mathbb{R}^{\{0\} \cup \mathcal{B}_\xi}$, $\pi_2^{(\xi)}(y) = y|_{\{0\} \cup \mathcal{B}_\xi}$.

Suppose that $f : X \times Y \rightarrow \mathbb{R}$ is a separately continuous function with $D(f) = E \times F$. Since X and Y are separable, f depends upon a countable quantity of coordinates, that is, there exist a $\xi \in [1, \omega_1)$ and a function $g : X_{\mathcal{A}_\xi} \times X_{\mathcal{B}_\xi} \rightarrow \mathbb{R}$ such that $f(x, y) = g(\pi_1^{(\xi)}(x), \pi_2^{(\xi)}(y))$ for every $x \in X$ and $y \in Y$. Note that mappings $\pi_1^{(\xi)}$ and $\pi_2^{(\xi)}$ are perfect, therefore g is a separately continuous function and $D(g) = E_\xi \times F_\xi$ by [7, Proposition 2]. Thus, $g \in \mathcal{F}_\xi$. Using (2) choose an $\eta \in [\xi, \omega_1)$ such that the sequences $(a_n^{(\eta)})_{n=1}^\infty$ and $(b_n^{(\eta)})_{n=1}^\infty$ nullify g in the coordinates $s_0 = 0$ and $t_0 = 0$.

Put $s_1 = A_\eta$, $t_1 = B_\eta$, $u_n = a_n^{(\eta)}$ and $v_n = b_n^{(\eta)}$ for every $n \in \mathbb{N}$. Clearly that the sequences $(u_n)_{n=1}^\infty$ and $(v_n)_{n=1}^\infty$ nullify f in the coordinates $s_0 = 0$ and $t_0 = 0$.

It follows from Namioka's theorem [1] that for the separately continuous functions $h_1 : E \times Y \rightarrow \mathbb{R}$, $h_1 = f|_{E \times Y}$, and $h_2 : X \times F \rightarrow \mathbb{R}$, $h_2 = f|_{X \times F}$, there exist dense in E and F respectively G_δ -sets $E_0 \subseteq E$ and $F_0 \subseteq F$ such that h_1 is jointly continuous at any point of $E_0 \times Y$ and h_2 is jointly continuous at any point of $X \times F_0$. Note that the sets $\{x \in E : x(s_1) = 1\}$ and $\{y \in F : y(t_1) = 1\}$ are open and nonempty in E and F respectively. Therefore there exist an $x_0 \in E_0$ and a $y_0 \in F_0$ such that $x_0(s_1) = 1$ and $y_0(t_1) = 1$.

Show that f is jointly continuous at (x_0, y_0) .

Fix $\varepsilon > 0$ and $k \in \mathbb{N}$ so that $\frac{1}{k} \leq \frac{\varepsilon}{2}$. Using the continuity of h_1 and h_2 at (x_0, y_0) choose $l \in \mathbb{N}$, $s_2, \dots, s_l \in \mathcal{A}$, $t_2, \dots, t_l \in \mathcal{B}$ and $\delta < \min\{u_k, v_k\}$ such that

$$|f(x, y) - f(x_0, y_0)| < \frac{\varepsilon}{2}$$

for every $(x, y) \in ((U \cap E) \times V) \cup (U \times (V \cap F))$, where $U = \{x \in X : x(0) < \delta, x(s_i) = x_0(s_i) \text{ for } 1 \leq i \leq l\}$ and $V = \{y \in Y : y(0) < \delta, y(t_i) = y_0(t_i) \text{ for } 1 \leq i \leq l\}$.

Show that $|f(x, y) - f(x_0, y_0)| < \varepsilon$ for every $x \in U$ and $y \in V$. Fix $x \in U$ and $y \in V$. Clearly that it is sufficient to consider the case of $x(0) > 0$ and $y(0) > 0$. Note that Proposition 4.2 implies $x(0) \in A_\eta$ and $y(0) \in B_\eta$. It follows from the choice of δ that there exist $n, m \geq k$ such that $x(0) = u_n$ and $y(0) = v_m$.

Assume that $m \geq n$. Since the function

$$\tilde{y}(t) = \begin{cases} 0, & \text{if } t = 0, \\ y(t), & \text{if } t \in \mathcal{B}, \end{cases}$$

belongs to $V \cap F$ and the sequences $(u_i)_{i=1}^\infty$ and $(v_i)_{i=1}^\infty$ nullify f in the coor-

dinates $s_0 = 0$ and $t_0 = 0$, we have

$$|f(x, y) - f(x, \tilde{y})| < \frac{1}{n} \leq \frac{1}{k} \leq \frac{\varepsilon}{2}.$$

Then

$$|f(x, y) - f(x_0, y_0)| \leq |f(x, y_0) - f(x, \tilde{y})| + |f(x, \tilde{y}) - f(x_0, y_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

If $n > m$, then we reason analogously.

Thus f is jointly continuous at (x_0, y_0) , which is impossible. \diamond

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